1) Let $f(x,y,z) = x^2 + y^2$.

$f_x = 2x$, $f_y = 2y$, $f_z = 0$.

so df is only 0 at $(x,y,z) = 0$.

So 1 is a regular value of f. Hence $f^{-1}(1) = \{(x,y,z) : x^2 + y^2 = 1\}$ is a regular surface. //

2) No, we cannot conclude $f^{-1}(0)$ is not a regular surface.
Consider the following example:

$f(x,y,z) = z^2$. Clearly $f^{-1}(0)$ is the xy plane and is a regular surface.

But $f_x = 0$, $f_y = 0$, $f_z = 2z$.

$\Rightarrow df$ vanishes at $(0,0,0)$, $\Rightarrow (0,0,0)$ is a critical point of f.

$\Rightarrow f(0,0,0) = 0$ is a critical value of f.

So we cannot conclude that $f^{-1}(0)$ is not a regular surface.

3) We show that $f: \mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism directly.

Let $p \in \mathbb{S}^2$. Then we find parameterizations

$X: U \subseteq \mathbb{R}^2 \to \mathbb{S}^2$ s.t. $p \in X(U)$, $p = X(u_0, v_0)$,

$Y: V \subseteq \mathbb{R}^2 \to \mathbb{S}^2$ s.t. $q = f(p) \in Y(V)$

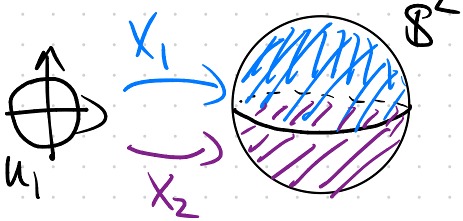
s.t. $(Y^{-1} \circ f \circ X): U \to V$ is differentiable at (u_0, v_0).

Take X, Y be the standard parameterization for the sphere.

Let $U_1 = \{(u,v) : u^2 + v^2 < 1\}$. Let $X_1 : U_1 \to \mathbb{S}^2$ by

$X_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$ and $X_2 : U_1 \to \mathbb{S}^2$ by

$X_2(u,v) = (u, v, -\sqrt{1-u^2-v^2})$



Then if p lies in the image of X_1,

$$f \circ X_1(u_0, v_0) = f(u_0, v_0, \sqrt{1-u_0^2-v_0^2}) = (-u_0, -v_0, -\sqrt{1-u_0^2-v_0^2}).$$

$$= (-u_0, -v_0, -\sqrt{1-(-u_0)^2-(-v_0)^2})$$

i.e. $f(p)$ lies in the image of X_2.

So $(X_2^{-1} \circ f \circ X_1)(u_0, v_0) = (-u_0, -v_0)$.

which is clearly differentiable as a map from $\mathbb{R}^2 \to \mathbb{R}^2$.

Similarly, if $p \in X_2(U_1)$, then $(X_1^{-1} \circ f \circ X_2)(u_0, v_0) = (-u_0, -v_0)$

is differentiable as a map from $\mathbb{R}^2 \to \mathbb{R}^2$.

So f is differentiable as a map from $S^2 \to S^2$.

Finally, observe that
$$f(f(x,y,z)) = f(-x,-y,-z) = (x,y,z),$$
so f is its own inverse.

So $f: S^2 \to S^2$ is a diffeomorphism. //.

4) Let $S = \{(x,y,z) : x^2+y^2=z\}$. Regarding $\mathbb{R}^2$ as a regular surface in $\mathbb{R}^3$, we show the existence of a diffeomorphism

Note we can parameterize S by $X: \mathbb{R}^2 \to S$ by

$$X(x,y) = (x, y, x^2+y^2).$$

Note that the plane has trivial parameterization Y by

$$Y(x,y) = (x,y)$$

Let $f: S \to \mathbb{R}^2$ be the map

$$f(x,y,z) = (x,y).$$

Then $(Y^{-1} \circ f \circ X)(x,y) = (Y^{-1} \circ f)(x,y,x^2+y^2)$

$$= Y^{-1}(x,y) = (x,y).$$

which is differentiable as a map $\mathbb{R}^2 \to \mathbb{R}^2$.

Let $g: \mathbb{R}^2 \to S$ by $g(x,y) = (x,y,x^2+y^2)$.

Then clearly $f \circ g = id = g \circ f$. So $g = f^{-1}$.

Moreover, $(X^{-1} \circ f^{-1} \circ Y)(x,y) = (X^{-1} \circ f^{-1})(x,y) = X^{-1}(x,y,x^2+y^2)$

$$= (x,y)$$

which is clearly also differentiable as a map from $\mathbb{R}^2 \to \mathbb{R}^2$.

So f is a diffeomorphism and S is diffeomorphic to $\mathbb{R}^2$. //

5) We parameterize S by $X(x, y, f(x,y))$, $p = (x_0, y_0, f(x_0,y_0))$.

Then $X_x = (1, 0, f_x)$, $X_y = (0, 1, f_y)$.

Since $T_pS = \text{span}\{X_u(p), X_v(p)\}$, we find the equation of the tangent plane by finding the normal, i.e. $X_x \times X_y$:

$$X_x \times X_y = (-f_x, -f_y, 1)$$

So the equation of the tangent plane at p, is given by

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z - f(x_0,y_0)) = 0.$$

$$\Rightarrow z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

as required.

Recall df_q is given by $\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \end{bmatrix}$

So $df_q((x,y)) = f_x(x_0,y_0)x + f_y(x_0,y_0)y$

and the graph of this function is given by

$$z = f_x(x_0,y_0)x + f_y(x_0,y_0)y.$$

which matches the equation of the tangent plane after translating to $(x_0, y_0, f(x_0,y_0))$.

/1.